

Multiplicative Lévy noise in bistable systems

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Stochastic motion in a bistable, periodically modulated potential is discussed. The system is stimulated by a white noise increments of which have a symmetric stable Lévy distribution. The noise is multiplicative: its intensity depends on the process variable like $|x|^{-\theta}$. The Stratonovich and Itô interpretations of the stochastic integral are taken into account. The mean first passage time is calculated as a function of θ for different values of the stability index α and size of the barrier. Dependence of the output amplitude on the noise intensity reveals a pattern typical for the stochastic resonance. Properties of the resonance as a function of α , θ and size of the barrier are discussed. Both height and position of the peak strongly depends on θ and on a specific interpretation of the stochastic integral.

I. INTRODUCTION

Ubiquity of the normal distribution in nature is a consequence of its stability. However, stable distributions are not restricted to the Gaussians: processes which involve discontinuous trajectories and algebraic tails $\sim |x|^{-1-\alpha}$, where $0 < \alpha < 2$ is a stability index, can be stable as well. Divergent moments of those distributions reflect a presence of long jumps (Lévy flights). The normal distribution corresponds to $\alpha = 2$. The Lévy stable statistics with $\alpha < 2$ is important when one considers realistic phenomena, where complexity, non-uniformity and long-range correlations play a role. For that reason, they are encountered and studied, for example, in biology [1], sociology [2] and finance [3, 4]. A characteristic feature of many problems in those fields is a non-homogeneous structure of the medium which results in an anomalous transport, as it is the case for porous, disordered materials [5] and folded polymers [6]. The non-homogeneity of the medium has significant consequences for a stochastic description of such systems in terms of the Langevin equation; the noise can no longer be additive and must include a dependence on the process variable. Despite that, multiplicative problems with Lévy flights are rarely discussed in the literature.

It is a well-known fact that the Langevin equation with the multiplicative white noise is not unique and requires an additional interpretation of the stochastic integral. Two of them are of particular interest: the Itô interpretation (II), for which the integrand is evaluated before the noise action, and the Stratonovich one (SI) which takes into account the process value both before and after the stochastic stimulation. SI is distinguished since it constitutes a limit of the coloured noise in a form of the fractional Ornstein-Uhlenbeck process [7]. On the other hand, II applies for discrete systems and systems with a large inertia [7, 8]. II leads to a fractional Fokker-Planck equation with a variable diffusion coefficient [9] and it differs from that for SI only by a spurious drift if $\alpha = 2$. For non-Gaussian processes a relation between both interpretations is more complicated and predictions qualitatively different: for SI, in contrast to II, slope of the distribution tail is modified by the multiplicative factor [10, 11]. In particular – in the case of the linear driving force – that factor may sufficiently weaken the noise to make the variance of the output signal finite. Slopes corresponding to the finite inertia lie between values for II and SI [7] which emphasises importance of taking into account both interpretations.

One also can achieve the finite variance by introducing a nonlinear deterministic force to a system with the additive noise [12]. From the physical point of view, the double-well potential is of particular interest since it allows us to study confined systems and, in particular, to model the barrier penetration. The double-well system was examined predominantly for the Gaussian case [13]; then the first passage time problem resolves itself to a dynamical equation with ordinary boundary conditions [14]. The barrier penetration for the Lévy flights was considered in Ref. [15, 16]. New effects emerge when one introduces – in addition to the double-well potential – a time-dependent periodic force. A rate of the jumping between the wells depends nonlinearly on the noise intensity and then a tuning of that intensity yields a matching of two characteristic frequencies: the noise-induced rate and a frequency of the deterministic, periodic force. That phenomenon, called a stochastic resonance [17], is well-known for the Gaussian noise [18] and was analysed also for the multiplicative case [19]. If the additive noise involves the Lévy flights the stochastic resonance also emerges; it was observed in systems with both the double- [20, 21] and single-well potential [22].

In this paper, we address a problem of the resonant switching between potential wells for the case of the multiplicative Lévy-stable stochastic stimulation. We demonstrate to what extent properties of the system are modified, in respect to the known cases, when one introduces long jumps and takes into account a dependence of the noise on position for both SI and II. In Sec.II the system, which includes the bistable potential and the oscillatory force, is defined and a mean first passage time calculated. Existence of the stochastic resonance is demonstrated, and dependence of the signal amplitude on the system parameters discussed, in Sec.III.

II. PROBABILITY DISTRIBUTIONS AND MEAN FIRST PASSAGE TIME

A general stable Lévy distribution is expressed in terms of three parameters: the stability index α ($0 < \alpha \leq 2$), which determines the tail of the distribution, the asymmetry parameter (skewness) and the translation parameter. Since, in the following, we restrict our analysis to the symmetric distributions, the index α completely determines the driving noise. The characteristic function for this case reads

$$\tilde{p}_\xi(k) = \exp(-\sigma^\alpha |k|^\alpha), \quad (1)$$

where σ measures the apparent width of the distribution. If $\alpha = 2$ Eq.(1) represents the normal distribution, otherwise the tails are algebraic, $\sim |\xi|^{-1-\alpha}$, and the stochastic trajectory is discontinuous in a sense of the Lindeberg condition [23]. The variance is infinite, as well as all higher moments.

We consider a motion in the time-dependent bistable potential

$$V(x, t) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 - A_0 x \cos(\omega_0 t) \quad (2)$$

where the amplitude A_0 and the frequency ω_0 are constant. The system is driven by the noise $G(x)\eta(t)$ and we assume that increments of the noise $\eta(t)$ are distributed according to the characteristic function (1). Then the Langevin equation is of the form

$$\dot{x} = -\partial V(x, t)/\partial x + G(x)\eta(t). \quad (3)$$

Intensity of the noise algebraically depends on the position,

$$G(x) = |x|^{-\theta}, \quad (4)$$

where $\theta > -1$. In this paper we take into account two interpretations of the multiplicative noise in Eq.(3): the Itô interpretation and the Stratonovich interpretation. Components of the discretized stochastic integral in II are defined by $G[x(t_{i-1})][\eta(t_i) - \eta(t_{i-1})]$, i.e. $G[x(t)]$ is calculated before the noise acts. In SI, in turn, one takes a middle point: $G[(x(t_{i-1}) + x(t_i))/2][\eta(t_i) - \eta(t_{i-1})]$. A difference between both interpretations, as well as probability density distributions for some nonlinear potentials, is well-known for the Gaussian processes [24, 25]. The Fokker-Planck equation, corresponding to Eq.(3) in II, involves a fractional derivative and a variable diffusion coefficient [9]:

$$\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} F(x) p(x, t) + \sigma^\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} |G(x)|^\alpha p(x, t), \quad (5)$$

where $F(x) = -\partial V(x, t)/\partial x$.

A technical advantage of SI consists in an admissibility of the variable change which allows us, in the one-dimensional case, to reduce Eq.(3) to the Langevin equation with the additive noise. Validity of the ordinary rules of the calculus was rigorously proved for $\alpha = 2$ [23]. It was confirmed for $\alpha < 2$ by the numerical simulations but the linear case required introduction of a cut-off to the distribution [10, 11]. On the other hand, rules of the ordinary calculus apply to systems with the coloured noise and the white-noise limit exists which was demonstrated for the generalised Ornstein-Uhlenbeck process [7]. After the transformation

$$y(x) = \frac{1}{1+\theta} |x|^{1+\theta} \text{sgn} x, \quad (6)$$

the Langevin equation takes the form

$$\dot{y} = -\partial \hat{V}(y, t)/\partial y + \eta(t) \quad (7)$$

where the effective potential reads

$$\hat{V}(y, t) = -\frac{a}{2}(1+\theta)y^2 + \frac{b}{2(2+\theta)}(1+\theta)^{2+2/(1+\theta)}y^{2+2/(1+\theta)} - \frac{A_0}{1+2\theta}(1+\theta)^{1+\theta/(1+\theta)}|y|^{1+\theta/(1+\theta)}\cos(\omega_0 t). \quad (8)$$

Wells of the time-independent part of the potential are positioned at $y_m = \pm(a/b)^{(1+\theta)/2}/(1+\theta)$ and their depth declines with θ .

Properties of the stochastic system are different for both interpretations of Eq.(3). The case $V(x, t) = 0$ was discussed in Ref.[10]. For II, the distribution tail does not depend on θ – it falls like $|x|^{-1-\alpha}$ – whereas SI leads to the dependence $|x|^{-1-\alpha-\theta\alpha}$. As a consequence, variance can be finite for SI. Result for the potential $V(x, t) = -\lambda x^2/2$, where $\lambda > 0$, is similar.

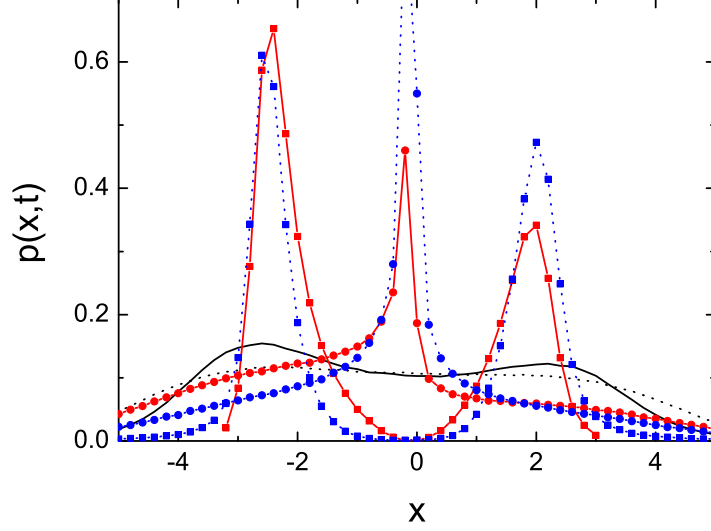


FIG. 1: (Colour online) The probability density distributions for II (blue dotted lines with symbols) and SI (red solid lines with symbols) calculated with $\alpha = 1.5$ at $t = 2$. The case $\theta = 2$ is marked by squares and $\theta = -0.5$ by circles. Other parameters are $a = 4$, $b = 1$, $A_0 = 3$, $\omega_0 = 5$ and $\sigma = 10$. The case $\theta = 0$ with $\alpha = 1.5$ and 2 are marked by the solid and dotted lines, respectively.

Presence of the nonlinear force modifies tails of the distribution, compared to linear case, making them steeper because, despite long jumps, the potential well prevents the particle from escaping to large distances. As a consequence, the variance may become finite. It is always the case for a quartic oscillator both with the additive noise [12] and the multiplicative noise in SI if $\theta > -1$ [11]. Motion in the double-well potential with the α -stable additive driving was considered by Ditlevsen [15]. The waiting and barrier penetration time – defined in terms of the reflection and/or absorbing barriers – is of particular interest for that system. An influence of the algebraic tails of the distribution for $\alpha < 2$ on the average transition time between the wells becomes visible if width of the barrier is sufficiently large; then the transition may be caused by a single jump [15]. Otherwise the motion is dominated by a continuous superposition of many small jumps, similarly to the Gaussian case, and the transition time depends exponentially on the potential depth according to the Arrhenius formula. The mean first passage time displays a bell-shape as a function of α : it rises with α up to $\alpha = 1$ and then falls [16]. In the case of the multiplicative noise, a behaviour of the transition time as a function of system parameters depends on the specific interpretation of the stochastic integral [11].

A further generalisation of the above system includes the periodic driving force. For this system, Eq.(3), we define the mean first passage time T_p as the average time needed to pass for the first time from the right minimum of the potential to the left one. Therefore, we introduce an absorbing barrier at $x = x_m = -\sqrt{a/b}$ in a sense that trajectories for which $x(t) \leq x_m$ are absorbed. The fractional Fokker-Planck equation corresponding to Eq.(3) involves a nonlocal boundary condition, since the jumping particle may skip the boundary without hitting it, and the problem cannot be solved analytically [16]. In the present paper, the stochastic trajectories were calculated by the numerical integration of Eq.(3). In the case of II, a simple Newtonian method was applied. For SI, in turn, we first transformed the variable $x \rightarrow y$, according to Eq.(6), and then applied the Heun method to the equation with the additive noise, Eq(7):

$$\begin{aligned} \tilde{y}_{i+1} &= y_i + h\hat{F}(y_i, t_i) \\ y_{i+1} &= y_i + \frac{h}{2}[\hat{F}(y_i, t_i) + \hat{F}(\tilde{y}_{i+1}, t_{i+1})] + h^{1/\alpha}\xi_i \end{aligned} \quad (9)$$

where $\hat{F}(y, t) = -\partial\hat{V}(y, t)/\partial y$.

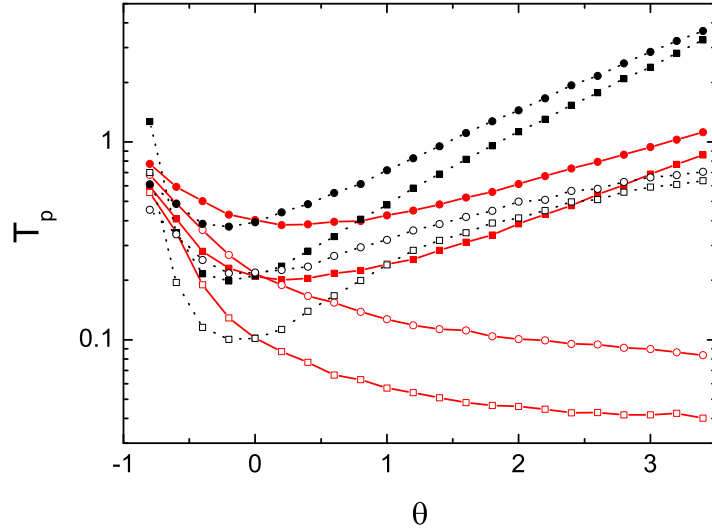


FIG. 2: (Colour online) Mean passage time from the left to the right well of the potential for II (black dotted lines) and SI (red solid lines) as a function of θ . The case $\alpha = 2$ is marked by squares and $\alpha = 1.5$ by circles. Other parameters are $a = 4$ (full symbols), $a = 1$ (empty symbols), $b = 1$, $A_0 = 3$, $\omega_0 = 5$ and $\sigma = 10$.

The multiplicative term $G(x)$ qualitatively modifies the distribution. First, we consider the case without the absorbing barrier. The distribution possesses two peaks, at negative and positive x , corresponding to the well positions. Their height decreases with σ and, for $\theta < 0$, they vanish altogether. Instead, a peak at the origin emerges as a result of increased contribution from the interwell motion. Fig.1 presents the probability density distributions obtained by a numerical integration of Eq.(3) with the initial condition $p(x, 0) = \delta(x - x_m)$. A relatively large noise intensity has been applied and the interwell motion prevails. In contrast to the case $\theta = 0$, the distribution is strongly anisotropic for both interpretations. Since the noise vanishes at the origin for $\theta < 0$, the particle abides near the top of the barrier. On the other hand, for the case $\theta > 0$ both peaks are present even for a large σ . Those conclusions apply for both II and SI. The case $\theta = 0$ (additive noise) exhibits small maxima at the well positions for $\alpha = 1.5$ whereas the Gaussian case is almost uniform in a wide range of the argument.

The mean passage time as a function of θ for various configurations is presented in Fig.2. Results depend both on the stability index α and on the particular interpretation of the stochastic integral. Moreover, a sensitivity on the potential geometry is visible. First, let us consider the case of the relatively wide barrier ($a = 4$). Depth of the effective potential for SI, Eq.(8), diminishes with θ and one could expect a smaller T_p . However, we know from Ref.[15] that, for a system with the additive noise, T_p depends rather on the width of the barrier than on the potential depth if jumps determine the barrier penetration. We observe a similar effect: position of the effective well reaches its minimum at $\theta_m = 2/\ln(a/b) - 1$ which value corresponds to the minimum of T_p in the figure. Nevertheless, there is no qualitative difference between the cases $\alpha = 1.5$ and 2. Shape of the curves $T_p(\theta)$ for II is similar but T_p rises faster for $\theta > 0$, as a result of a strong noise damping. The passage time as a function of θ for $a = 1$ is different than for $a = 4$ because then $y_m(\theta) = 1/(1 + \theta)$, θ_m does not exist and $T_p(\theta)$ monotonically falls. Since $\theta > -1$, such a behaviour is expected for all narrow barriers, namely if $a \leq b$. Obviously, T_p in Fig.2 decreases with α because the noise intensity $\sim \sigma^\alpha$.

III. STOCHASTIC RESONANCE

The rate of jumping between potential wells due to the noise can match the frequency ω_0 of the deterministic oscillatory force revealing a resonant pattern. This phenomenon has been extensively studied for the Gaussian

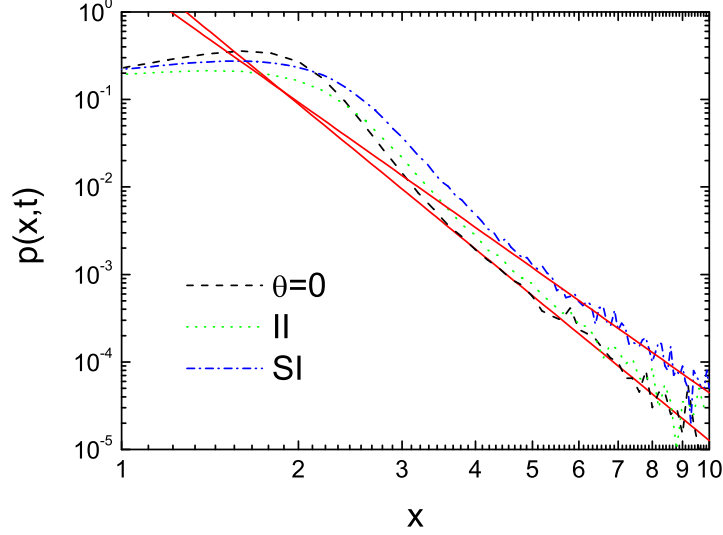


FIG. 3: (Colour online) The probability density distributions for both interpretations calculated at $t = 2$ with the parameters: $\alpha = 1.5$, $a = 4$, $b = 1$, $A_0 = 3$, $\omega_0 = 5$ and $\sigma = 2$. Cases with the multiplicative noise were calculated with $\theta = -0.5$. The red solid lines are of the form $x^{-\mu}$ with $\mu = 5.5$ and 4.75 .

processes and observed also for the general Lévy statistics of the noise. It was demonstrated in Ref.[20] that the signal-to-noise ratio exhibits a resonant amplification for many systems characterised by the long jumps. The spectral amplification and dynamic hysteresis loops were calculated for the asymmetric Lévy noise with various α and skewness parameter [21]; the amplification dwindles with decreasing α . On the other hand, the stochastic resonance phenomenon for a linear multiplicative Gaussian noise was studied in Ref.[19] and attributed to the transition from an intrawell to an interwell modulated dynamics.

A useful quantity to describe a stochastic resonance is a correlation between the output process $x(t)$ and the periodic driving, \bar{x} . It equals the first Fourier coefficient which follows from the expression [26]

$$M_n = \frac{\omega_0}{\pi} \int_{-\infty}^{\infty} dx \int_0^T xp(x, t) \exp(in\omega_0 t) dt, \quad (10)$$

where $T = 2\pi/\omega_0$ and $p(x, t)$ results from Eq.(3). The density $p(x, t)$ is asymptotically periodic and an averaging over the period produces a stationary distribution. The spectral amplification at the frequency ω_0 is given by a ratio of the integrated power stored in spikes of the power spectrum to the total power carried by the oscillatory force, $\eta = 4(\bar{x}/A_0)^2$, where $\bar{x} = \text{Re}M_1$ [18]. The signal-to-noise ratio (SNR) also can be expressed by M_1 : $\text{SNR} = 4\pi|M_1|^2/S_N^0(\omega_0)$ where S_N^0 is the total power spectral density. The latter quantity is directly related to the autocorrelation function and its existence requires the finite variance. That condition is satisfied for systems with the quartic potential in SI since then the tail $\sim |x|^{-3-\alpha-\alpha\theta}$ [11]. On the other hand, an analysis for both the free particle and the linear force indicates that in II shape of the distribution tails does not depend on θ . Fig.3 shows that also for the general system (3) the tails fall faster than for the simple Lévy motion: the slope rises with θ for SI whereas it does not depend on θ for II. Therefore variance remains finite when Lévy flights are included and the spectral quantities, η and SNR, are finite and well determined.

For small amplitudes and large time, a response of the system to the periodic stimulation is proportional to \bar{x} , $\langle x(t) \rangle_{as} = \bar{x} \cos(\omega_0 t - \bar{\phi})$, where $\bar{\phi}$ is a phase lag [18]. The stochastic resonance is usually detected by plotting \bar{x} as a function of the stochastic stimulation σ . That function does not monotonically diminish but exhibits a bell-shaped structure maximum of which corresponds to a matching condition of two frequencies: ω_0 and a rate of the stochastic jumping, $1/T_p$. Then we can expect that a position of the resonance is directly related to the mean passage time and satisfies the approximate condition

$$\omega_0 T_p(\sigma, \alpha, \theta) \sim 1. \quad (11)$$

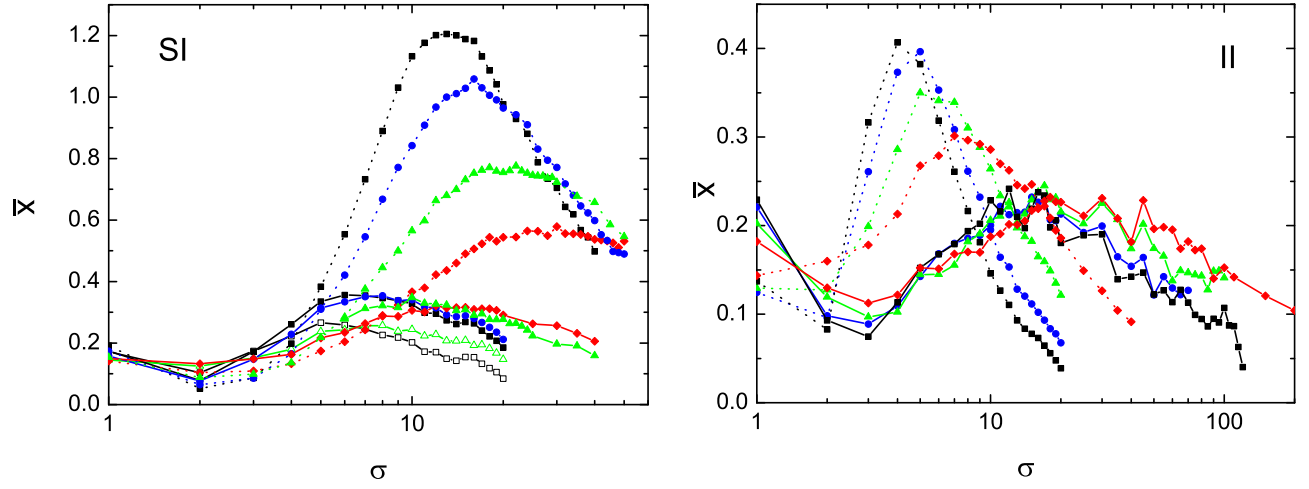


FIG. 4: (Colour online) \bar{x} as a function of σ for $\theta = 1.1$ (solid lines) and -0.6 (dotted lines). The following values of α are presented: 2 (black squares), 1.8 (blue circles), 1.5 (green triangles) and 1.2 (red diamonds). Lines with empty symbols corresponds to $\theta = 0$. Other parameters are $a = 4$, $b = 1$, $A_0 = 3$, $\omega_0 = 5$. Averaging was performed over 2×10^4 events.

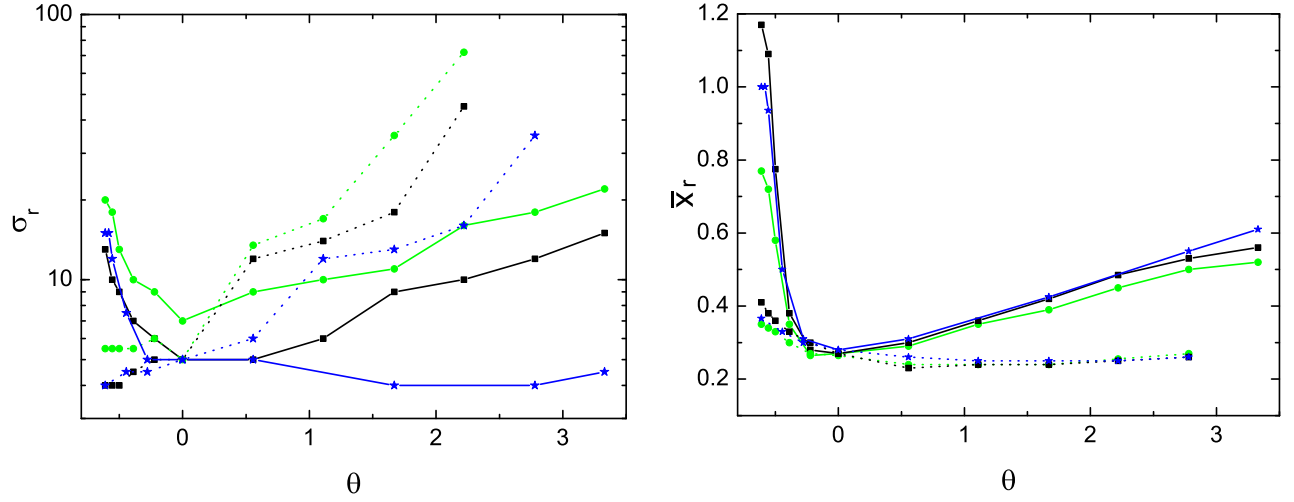


FIG. 5: (Colour online) Left part: σ_r versus θ for SI (solid lines) and II (dotted lines). The case $a = 4$ was calculated with $\alpha = 2$ (black squares) and 1.5 (green circles), the case $a = 1$ with $\alpha = 1.8$ (blue stars). Other parameters are the same as in Fig.4. Right part: height of the peak versus θ .

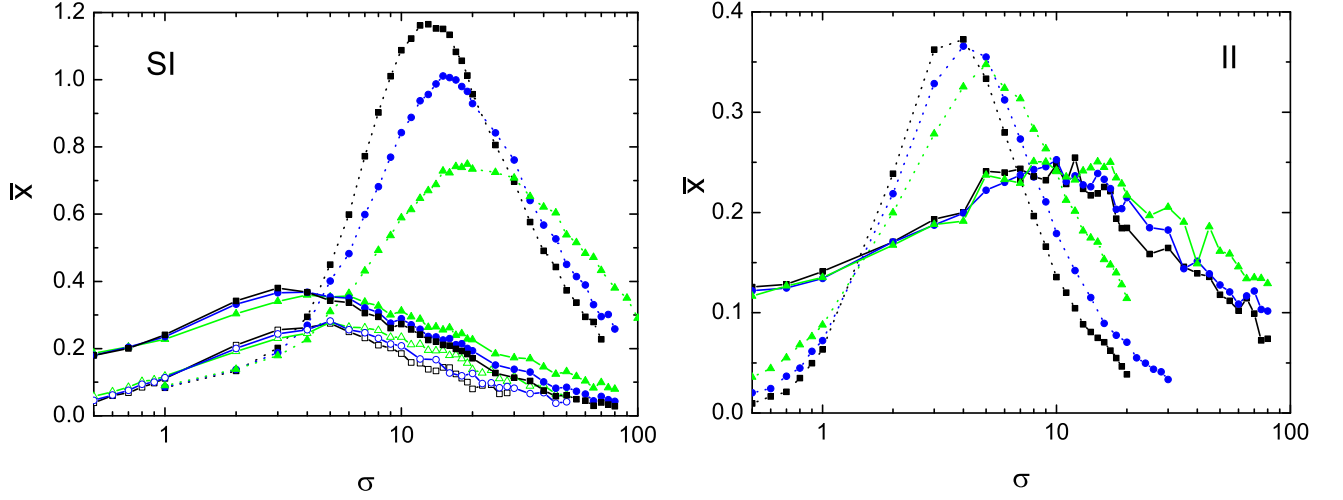


FIG. 6: (Colour online) The same as Fig.4 but for $a = 1$ and $\alpha = 2, 1.8$ and 1.5 .

Height of the peak rises with the amplitude A_0 and diminishes with the frequency ω_0 . In the following, we discuss an emergence of the stochastic resonance in the system (3) and demonstrate how its position and intensity depends on α , θ , as well as geometry of the potential.

Fig.4 presents results of the calculations: a dependence of \bar{x} on σ for two values of θ , both positive and negative; comparison with the additive noise is included. The resonance structure is visible for all the presented cases. Remarkable differences emerge when one compares both interpretations. For SI, \bar{x} is large if $\theta < 0$ and a position of the peak, σ_r , occurs at larger noise intensity than for $\theta > 0$ since then the effective barrier is wider. The resonance for the positive θ emerges at roughly the same σ as for $\theta = 0$. The peak is much lower for II if $\theta < 0$ and corresponds to a smaller σ than for $\theta > 0$. The latter case, governed by Eq.(5), is characterised by a weak noise intensity when the particle remains far from the origin and that effect must be compensated by a larger σ . Dependence on α is similar for both interpretations: σ_r shifts towards smaller σ and $\bar{x}_r = \bar{x}(\sigma_r)$ rises with α for $\theta < 0$. σ_r falls with α also for $\theta > 0$ but this effect is much weaker, especially for II. A dependence of σ_r on θ is presented in Fig.5 for $\alpha = 1.5$ and 2. The difference between both interpretations is especially pronounced for $\theta < 0$ when σ_r remains constant for II, in contrast to SI. For $\theta > 0$, all presented cases exhibit a monotonically increasing function $\sigma_r(\theta)$. Height of the resonance, \bar{x}_r , is also presented in Fig.5 as a function of θ . This dependence is very weak for II, in contrast to SI for which we observe a pattern similar to $\sigma_r(\theta)$ with a minimum at the same point. Results for both values of α are almost the same. An apparent similarity between Fig.2 and 5, which is especially striking for SI, is a consequence of the condition (11); since the curves $T_p(\theta)$ for different σ are, approximately, parallel to each other, they mimic the dependence $\sigma_r(\theta)$.

A change of the potential shape qualitatively modifies T_p and one can expect differences also for the stochastic resonance. This conclusion is illustrated in Fig.5 which compares the case of the wide barrier ($a = 4$) with the case of the narrow barrier ($a = 1$) for both interpretations. Though results for II do not differ substantially – σ_r rises for both geometries – the curve for SI and $a = 1$ is flat resembling a similar behaviour of T_p , cf. Fig.2. Comparison of the peak heights is also presented in Fig.5. We conclude that the amplitude is not sensitive not only to α but also to the barrier width, in contrast to peaks' position. The amplification is much stronger for SI than for II and the dependence $\bar{x}_r(\theta)$ is determined mainly by a specific interpretation of the stochastic integral. Curves $\bar{x}(\sigma)$ for the narrow barrier, shown in Fig.6, are similar to the case $a = 4$ (Fig.4); difference resolves itself to a smaller σ_r for SI if $\theta \geq 0$.

IV. CONCLUSIONS

Properties of systems containing stochastic force, which is characterised by the stable distribution with the heavy tails, differ from those for the normal distribution possessing, in particular, the infinite variance. Systems with the multiplicative noise exhibit a more complicated behaviour and the variance may be finite, despite the algebraic tails. Nonlinear potentials, in turn, makes the distribution tails steeper and may invoke an effective trapping. In such systems motion is limited in space and asymptotics of the driving noise plays a relatively minor role, compared to the diffusion problems. Motion in the double-well potential is a simple example.

We have discussed such a system: the anharmonic oscillator with the periodic modulation and the stochastic stimulation in a form of the stable, multiplicative process. The multiplicative factor, assumed in the algebraic form, makes the distribution anisotropic; the point $x = 0$ becomes an attractor for $\theta < 0$ since noise vanishes at the origin. The mean first passage time depends on a particular interpretation of the stochastic integral; for SI it is determined rather by the size of the effective potential than by the effective well depth. That conclusion, however, is valid only for a relatively wide barrier and then the geometry is important for the dependence $T_p(\theta)$. The passage time smoothly rises with α for both geometries and no qualitative difference in respect to $\alpha = 2$ has been observed.

The system (3) possesses two characteristic frequencies, ω_0 and $1/T_p$, and, when they match, the resonance occurs. The resonance shape of the dependence $\bar{x}(\sigma)$ has been observed for all analysed cases. Since the variance is finite also for processes with the Lévy flights, both the spectral amplification η and SNR are well determined. For SI, height of the resonance is sensitive on θ and the amplification rapidly rises when θ becomes negative. Position of the resonance, in turn, mimics the dependence $T_p(\theta)$. Therefore, size of the barrier is important for the resonance position: σ_r is almost constant for SI if $a = 1$, in contrast to the case $a = 4$. We finally conclude that – though height and position of the peaks depend on α – the bistable system with the Lévy flights has similar properties as the case of the Gaussian processes. On the other hand, dependence of the noise on the process value essentially modifies – differently for each interpretation of the stochastic integral – the distributions, the passage time and properties of the stochastic resonance.

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